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On the Plane-Width of Graphs

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Abstract

Map vertices of a graph to (not necessarily distinct) points of the plane so that two adjacent vertices are mapped at least a unit distance apart. The *plane-width* of a graph is the minimum diameter of the image of the vertex set over all such mappings. We establish a relation between the plane-width of a graph and its chromatic number, and connect it to other well-known areas, including the circular chromatic number and the problem of packing unit discs in the plane.

Keywords: plane-width, realization of a graph, chromatic number, circular chromatic number

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1 Introduction

This is an extended abstract of [6]. Given a simple, undirected, finite graph $G = (V, E)$, a *realization* of G is a function r assigning to each vertex a point in the plane such that for each $\{u, v\} \in E$, $d(r(u), r(v)) \geq 1$, where d is the Euclidean distance. The *width* of a realization is the maximum distance between the images of any two vertices. In this paper, we introduce a new graph invariant, called the **plane-width** and denoted by $pw(G)$, which is the minimum width over all realizations of G. (To avoid trivialities we only consider graphs with at least one edge.)

Complete graphs. The problem of determining the plane-width of complete graphs K_n has previously appeared in the literature in different contexts: finding the minimum diameter of a set of n points in the plane such that each pair of points is at distance at least one [2], or packing non-overlapping unit discs in the plane so as to minimize the maximum distance between any two disc centers [7]. The exact values of $pw(K_n)$ are known only for complete graphs on at most 8 vertices. However, the asymptotic behaviour of $pw(K_n)$ has been determined.

Theorem 1.1 ([1,2,5])

$$\lim_{n\to\infty} pw(K_n)/\sqrt{n} = \left(2\pi^{-1}3^{1/2}\right)^{1/2} \approx 1.05\,.$$

The plane-width of K_4 is $\sqrt{2}$ and our first result is a generalization of this fact.

Proposition 1.2 *The plane-width of every odd wheel is equal to $\sqrt{2}$.*

2 Plane-width and chromatic number

Small chromatic number. For small values of the chromatic number, there is a strong relation between the plane-width of a graph and its chromatic number.

Theorem 2.1 *For all graphs G,*

(a) $pw(G) = 1$ if and only if $\chi(G) \leq 3$,

(b) $pw(G) \notin (1\,, 2/\sqrt{3}]$,

(c) $pw(G) \in (2/\sqrt{3}\,, \sqrt{2}]$ if and only if $\chi(G) = 4$,

(d) $pw(G) \in (\sqrt{2}\,, 2]$ if and only if $\chi(G) \in \{5, 6, 7\}$.

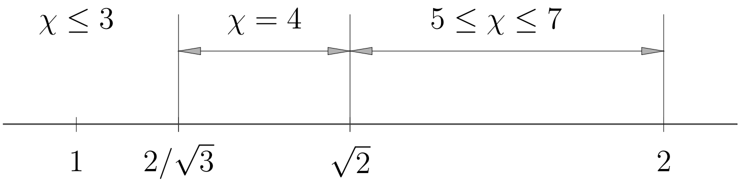

Fig. 1. Relation between pw and χ for small values of these invariants.

In particular, every bipartite graph has plane-width exactly 1. Also, every graph with maximum degree at most 3, different from the complete graph on 4 vertices, has plane-width exactly 1. (By Brooks's Theorem such graphs are 3-colorable.) The plane-width of every planar graph is at most $\sqrt{2}$ (as such graphs are 4-colorable), and the plane-width of graphs embeddable on a torus is at most 2 (as such graphs are 7-colorable).

Large chromatic number. We have already seen in Theorem 1.1 that $pw(K_n) = \Theta(\sqrt{n})$. We show, more generally, that the relation $pw(G) = \Theta(\sqrt{\chi(G)})$ holds for arbitrary graphs as $\chi(G) \to \infty$.

Lemma 2.2 *For every $\epsilon > 0$ there exists an integer k such that for all graphs G of chromatic number at least k, it holds that $\chi(G) < \left(\left(\frac{2}{\sqrt{3}} + \epsilon\right) \cdot pw(G)\right)^2$.*

Lemma 2.3 *For all graphs G, $pw(G) \leq pw(K_{\chi(G)})$.*

The two lemmas give a lower and an upper bound which are combined in the following theorem.

Theorem 2.4 *For every $\epsilon > 0$ there exists an integer k such that for all graphs G of chromatic number at least k,*

$$\left(\frac{\sqrt{3}}{2} - \epsilon\right)\sqrt{\chi(G)} < pw(G) < \left(\sqrt{\frac{2\sqrt{3}}{\pi}} + \epsilon\right)\sqrt{\chi(G)}\,.$$

Some questions regarding the plane-width of a graph can be answered via the chromatic number by applying Theorem 2.4. For instance, the plane-width of almost every random graph (in the $G_{n,p}$ model with a fixed $p \in (0,1)$) is $\Theta(\sqrt{n/\log(n)})$ (since the chromatic number of almost every random graph is $\Theta(n/\log(n))$ [3]). Another example is the existence of graphs of arbitrarily large plane-width and girth (as there are graphs of arbitrarily large chromatic number and girth [4]).

Open problem. *Let $\mathbb{P} = \{pw(G) : G$ is a graph$\}$. Determine whether there exists a function (a non-decreasing function) $f : \mathbb{P} \to \mathbb{Z}$ such that $f(pw(G)) = \chi(G)$ for every non-bipartite graph G.*

3 Plane-width and circular chromatic number

The circular chromatic number $\chi_c(G)$ is a well-known graph invariant and can be seen as a refinement of the chromatic number. We establish a connection between the circular chromatic number and the plane-width.

Lemma 3.1 *For all graphs G, $pw(G) \leq \left[\sin\left(\frac{\pi}{\chi_c(G)}\right)\right]^{-1}$.*

This allows us to apply some known results on the circular chromatic number to prove the existence of graphs with certain plane-widths. Specifically, we obtain the following theorem, which should be viewed as complementing Theorem 2.1.

Theorem 3.2 *For every $\epsilon > 0$ there exists*

(a) A 4-chromatic graph G such that $pw(G) < 2/\sqrt{3} + \epsilon$,

(b) A 5-chromatic graph G such that $pw(G) < \sqrt{2} + \epsilon$,

(c) An 8-chromatic graph G such that $pw(G) < 2 + \epsilon$.

4 Plane-width and graph operations

Homomorphisms and perfect graphs. Any graph with chromatic number $\chi(G)$ is homomorphic to $K_{\chi(G)}$. The following lemma generalizes Lemma 2.3.

Lemma 4.1 *Let G be a graph homomorphic to a graph H. Then, $pw(G) \leq pw(H)$.*

We denote by $\omega(G)$ the maximum size of a clique in G.

Corollary 4.2

(a) For every graph G and its subgraph G', $pw(G') \leq pw(G)$.

(b) For every graph G, $pw(G) \geq pw(K_{\omega(G)})$.

These observations together with Lemma 2.3 imply that for graphs whose chromatic number coincides with their maximum clique size, their plane-width is a function of their chromatic number.

Corollary 4.3 *Let G be a graph such that $\chi(G) = \omega(G)$. Then, $pw(G) = pw(K_{\chi(G)})$. In particular, if G is perfect, then $pw(G) = pw(K_{\chi(G)})$.*

Cartesian product. Let $G\Box H$ be the Cartesian product of G and H. Corollary 4.2 implies that $pw(G\Box H) \geq \max\{pw(G), pw(H)\}$. In the following theorem, we provide an exact and an asymptotic upper bound on $pw(G\Box H)$.

Theorem 4.4

(a) *For every two graphs G and H,*

$$pw(G\Box H) \leq pw(G) + pw(H)\,.$$

(b) *For every $\epsilon > 0$ there exists a $p > 0$ such that for every two graphs G and H of plane-width at least p,*

$$pw(G\Box H) \leq \left(\sqrt{\frac{8}{\sqrt{3}\pi}} + \epsilon\right) \max\{pw(G), pw(H)\}\,.$$

Disjoint union. Let $G \uplus H$ denote the disjoint union of G and H. By Corollary 4.2, we have $pw(G \uplus H) \geq \max\{pw(G), pw(H)\}$.

Theorem 4.5 *For every two graphs G and H, we have that*

$$pw(G \uplus H) \leq \max\left(pw(G), pw(H), \frac{1}{\sqrt{3}}(pw(G) + pw(H))\right)\,.$$

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